Analysis of Stability (2+1)-Dimensional Fuzzy Navier-Stokes Equations Based on Lipschitz Condition

Zena Talal Yassin

Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Nineveh- Iraq.

Corresponding author: zena-talal@uomosul.edu.iq

Abstract:

We apply a rigorous stability analysis to the two-dimensional fuzzy Navier-Stokes equations that are formulated under the α-cut representation. For this study, we used the nonlinear advection-pressure operator to test the Lipschitz condition of the fuzzy velocity field on the initial fuzzy data in order to prove its continuous dependence. The fuzzy version of the Navier-Stokes model introduces uncertainty in fundamental physical parameters including viscosity, density, and external forces. We express the system in abstract operator form, while also arriving at a differential inequality that governs the evolution of perturbation between two fuzzy solutions. By Lipschitz continuity and Grönwall's inequality, the differential between any two fuzzy velocity solutions is shown to be bounded by an exponential function of time. Hence, it is demonstrated that at every α-level the fuzzy Navier-Stokes system is stable, such that the overall fuzzy solution will inherit the stability property. This provides a theoretical foundation for analyzing uncertain or imprecise fluid dynamics models, which can be expanded to energy stability or numerical verification in further research.

Keywords: Fuzzy Navier–Stokes equations; Stability analysis, α-cut representation, Lipschitz continuity, Grönwall inequality, Fuzzy partial differential equations, Uncertain fluid dynamics.

1. Introduction

The Navier–Stokes equations have proved a cornerstone in the determination of the movement of viscous incompressible fluids, and form the backbone of today's fluid dynamics. These equations account for several phenomena in physics: laminar and turbulent flows, heat transfer and aerodynamics. In the classical deterministic form all parameters – viscosity, density and external forces – are assumed to be perfectly known. Nonetheless, in actual fluid systems such parameters are generally uncertain and can also be affected by measurement error, environmental fluctuations or intrinsic system variability [1-5]. For such uncertainties, fuzzy set theory (Zadeh 1965) offers a powerful mathematical tool for portraying imprecise or vague information. Incorporating the Fuzzy concepts into differential equations has resulted in the development of Fuzzy differential and Fuzzy partial differential equations (FPDE)[13-14]. These formulations open the research to physical systems that have non deterministic governing parameters and can be described using Fuzzy numbers or Fuzzy functions. In this context, the Fuzzy Navier-Stokes equations often a means of applying fluid flow under uncertain or imprecise physical conditions[6-9].

Despite the extensive literature on deterministic and stochastic version of the Navier-Stokes equations, the analysis of their fuzzy counterparts remains relatively derexplored. In particular, while several studies have focused on numerical approximations and fuzzy modeling of fluid dynamics, few have provided rigorous analytical proofs of the stability of fuzzy solutions. Stability analysis is essential since ensures that small perturbations in the fuzzy initial or boundary data lead to proportionally small deviations in the resulting fuzzy velocity and pressure fields[10-12].

The objective of this paper is to present a detailed mathematical proof of the stability of the two-dimensional fuzzy Navier-Stokes equations under the α −cut representation. By reformulating the system in operator form and assuming that the nonlinear advection-pressure operator satisfies a global Lipschitz condition, we establish a differential inequality that governs the evolution of the fuzzy velocity perturbation. The application of Gromwell's inequality than leads to an exponential stability estimate[15-17].

The remainder of this paper is organized as follows; Section 2 presents the basic mathematical definitions and preliminaries related to fuzzy numbers, α −cut representation

In this work, two spatial dimensions plus one time dimension are referred to as (2+1)-dimensional.

and the Lipschitz condition, Section 3 formulates the fuzzy Navier-Stokes model and introduces the operator framework. Section 4 provides the main stability theorem and its proof using the Lipschitz condition. Section 5 discusses the implications of the obtained result and possible numerical extensions. Finally, Section 6 concludes the paper with remarks on potential directions for future research.

2. Mathematical Preliminaries

In this suction, we introduce the fundamental concepts and definitions that form the basis for the analysis of fuzzy Navier-Stokes equations.

2.1. Fuzzy numbers [18]

A fuzzy number $\tilde{x}$ is a convex, normalized fuzzy set on the real line $\mathbb{R}$ with a continuous membership function $\mu_{\tilde{x}}: \mathbb{R} \to [0,1]$ where $\mu_{\tilde{x}}(\tilde{x})$ represents the degree of membership of x in $\tilde{x}$. A fuzzy number satisfies the following properties:

- There exists a point x_0 such that $\mu_{\tilde{x}}(x_0) = 1$ (normality).
- $\mu_{\tilde{x}}(x)$ is upper semicontinuous.
- The support $supp(\tilde{x}) = \{x \in \mathbb{R} | \mu_{\tilde{x}}(x) > 0\}$ is bounded.

2.2. α −Cut Representation [19]

For a fuzzy number $\tilde{x}$, the α −cut $\tilde{x}_\alpha$ is defined as the set of all real numbers with membership degree at least $\alpha \in [0,1]$.

$$\tilde{x}_\alpha = \{x \in \mathbb{R} | \mu_{\tilde{x}}(x) \geq \alpha\}$$

Each α −cut is a closed interval:

$$\tilde{x}_\alpha = [x_\alpha^L, x_\alpha^U]$$

Where x_α^L and x_α^U denote the lower and upper bounds of the α −level interval. The α −cut representation allows the fuzzy system to be treated as a family of interval-valued deterministic systems at each α −level.

2.3. Fuzzy Functions and Fuzzy Partial Derivatives

A function $\tilde{u}: \Omega \times [0,T] \to \mathbb{F}(\mathbb{R})$ is fuzzy-valued if for each $(x,y,t) \in \Omega \times [0,T]$, $\tilde{u}(x,y,t)$ is a fuzzy number[20].

The partial derivative of a fuzzy function $\tilde{u}$ with respect to x at level α is defined via α −cuts:

$$\frac{\partial \tilde{u}_\alpha}{\partial x} = \left[\frac{\partial u_\alpha^L}{\partial x}, \frac{\partial u_\alpha^U}{\partial x}\right]$$

Similarly, the derivatives and high-order derivatives are defined component wise at each α −level.

2.4. Functional Spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. For the analysis, we use standard Hulbert and Sobolev spaces:

- $L^2(\Omega)$: Space of square-integrable functions over Ω with inner product $\langle u, v\rangle = \int_\Omega u(x)v(x)dx$ and norm $\|u\| = \sqrt{\langle u, u\rangle}$.
- $H^1(\Omega)$: Space of functions with square-integrable first derivatives.
- $H_\Omega^1(\Omega)$: Subspace of $H^1(\Omega)$ with zero boundary values.

These spaces allow the rigorous formulations of the fuzzy Navier-Stokes equations and the application of operator-theoretic methods[21].

2.5. Lipschitz Continuity

Let $\mathcal{N}_\alpha: H \to H$ be a nonlinear operator at α −level. It satisfies a Lipschitz condition if there exists a constant $L_\alpha > 0$ such that for all $u_1, u_2 \in H$:

$$\|\mathcal{N}_\alpha(u_1) - \mathcal{N}_\alpha(u_2)\| \leq L_\alpha \|u_1 - u_2\|$$

This property is Crucial for proving the continuous dependence of fuzzy solutions on initial data and establishing exponential stability via Grönwall's inequality[22].

2.6. Grönwall's Inequality

Let $\phi(t)$ be a non-negative continuous function on $[0, T]$ and suppose it satisfies:

$$\phi(t) \leq C + \int_0^t \beta(s)\phi(s)ds$$

Where $C \geq 0$ and $\beta(t)$ is integrable. The Grönwall's inequality implies:

$$\phi(t) \leq C \exp\left(\int_0^t \beta(s)ds\right), \qquad \forall t \in [0, T]$$

The inequality will be applied to bound the norm of perturbation between two fuzzy solutions[23].

3. Classical Navier-Stokes equations

$$\nabla \cdot u = 0$$

$$\frac{\partial u}{\partial t} + (\nabla \cdot u)u = -\frac{1}{\rho}\nabla p + v\nabla^2 u + f$$

$u(x,t)$: velocity vector field

$p(x,t)$: pressure

ρ: fluid density

v: kinematic viscosity

f: external body forces (e.g., gravity)

Fuzzy Navier-Stokes equations (uncertain/fuzzy form) in the fuzzy formulation, physical quantities are represented by fuzzy numbers or fuzzy functions, to model uncertainty or imprecision in measurements such as velocity, viscosity and density[24], we define fuzzy fields as $\tilde{u}(x,t,\alpha)$, $\tilde{p}(x,t,\alpha)$, $\tilde{\rho}(\alpha)$, and $\tilde{v}(\alpha)$, where $\alpha \in [0,1]$ is the membership level (α −cut) representing the degree of confidence or behaving, then the fuzzy Navier-Stokes equations became:

$$\nabla \cdot \tilde{u}(x,t,\alpha) = 0$$

$$\frac{\partial \tilde{u}(x,t,\alpha)}{\partial t} + (\tilde{u}(x,t,\alpha) \cdot \nabla)\tilde{u}(x,t,\alpha) = -\frac{1}{\tilde{\rho}(x)}\nabla\tilde{p}(x,t,\alpha) + \tilde{v}(\alpha)\nabla^2\tilde{u}(x,t,\alpha) + \tilde{f}(x,t,\alpha)$$

2D steady fuzzy Navier-Stokes equations[12]:

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{1}{\tilde{\rho}}\frac{\partial \tilde{p}}{\partial x} + \tilde{v}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right)$$

$$\tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} = -\frac{1}{\tilde{\rho}}\frac{\partial \tilde{p}}{\partial y} + \tilde{v}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right)$$

4. Main Theorem

Theorem (2D fuzzy stability / continuous dependence)

Proof of stability for the 2D fuzzy Navier-Stokes equations using the Lipschitz condition:

We consider the incompressible 2D fuzzy Navier-Stokes equations in α −cut representation:

$$\nabla \cdot \tilde{u}_\alpha = 0$$

$$\frac{\partial \tilde{u}_\alpha}{\partial t} + (\nabla \cdot \tilde{u}_\alpha)\tilde{u}_\alpha = -\frac{1}{\tilde{\rho}_\alpha}\nabla \tilde{p}_\alpha + \tilde{v}_\alpha \nabla^2 \tilde{u}_\alpha - \tilde{f}_\alpha$$

Where $\tilde{u}_\alpha(x, y, t) = (\tilde{u}_\alpha, \tilde{v}_\alpha)$ is the fuzzy velocity field at α –level, $\tilde{p}_\alpha$ is fuzzy pressure, $\tilde{v}_\alpha$, $\tilde{\rho}_\alpha$ are fuzzy viscosity and density parameters and all fuzzy parameters are continuous and bounded for $\alpha \in [0,1]$.

For each α –level, we can write the system in abstract operator form on Hilbert space:

$$H = L^2(\Omega)^2$$

$$\frac{\partial \tilde{u}_\alpha}{\partial t} = A_\alpha \tilde{u}_\alpha + N_\alpha(\tilde{u}_\alpha) + \tilde{f}_\alpha$$

Where $A_\alpha \tilde{u}_\alpha = \tilde{v}_\alpha \nabla^2 \tilde{u}_\alpha$ (linear differential operator), $N_\alpha(\tilde{u}_\alpha) = -\tilde{u}_\alpha\,(\nabla \cdot \tilde{u}_\alpha) - \frac{1}{\tilde{\rho}_\alpha}\nabla \tilde{p}_\alpha$ (nonlinear advection-pressure parameter).

Assume that the nonlinear operator $N_\alpha(\cdot)$ satisfies Lipschitz condition in H:

$$\| N_\alpha(u_1) - N_\alpha(u_2) \| \le L_\alpha \| u_1 - u_2 \|$$

For all $u_1, u_2 \in H$, where $L_\alpha > 0$ is the Lipschitz constant depending on α but uniformly bounded, this means small perturbations in the velocity field lead to proportionally small changes in the nonlinear term.

Let $\tilde{u}_\alpha(x, y, t)$ and $\tilde{v}_\alpha(x, y, t)$ be two fuzzy velocity solutions corresponding to slightly different fuzzy initial date $\tilde{u}_\alpha(x, y, 0) = \tilde{u}_{0,\alpha}$, $\tilde{v}_\alpha(x, y, 0) = \tilde{v}_{0,\alpha}$.

Define the difference:

$$w_\alpha = \tilde{u}_\alpha - \tilde{v}_\alpha$$

We say the solution is stable if:

$$\| w_\alpha(t) \| \le C_\alpha \| w_\alpha(0) \|, \qquad \forall t \ge 0$$

Where C_α is a bounded constant

Subtracting the two fuzzy systems gives:

$$\frac{dw_\alpha}{dt} = A_\alpha w_\alpha + [N_\alpha(\tilde{u}_\alpha) - N_\alpha(\tilde{v}_\alpha)]$$

Take the L^2 inner product with w_α:

$$\langle\frac{dw_\alpha}{dt}, w_\alpha\rangle = \langle A_\alpha w_\alpha, w_\alpha\rangle + \langle N_\alpha(\tilde{u}_\alpha) - N_\alpha(\tilde{v}_\alpha), w_\alpha\rangle$$

The left-hand side gives:

$$\frac{1}{2}\frac{d}{dt}\|w_\alpha\|^2$$

For the viscosity term, using integration by parts and incompressibility:

$$\langle A_\alpha w_\alpha, w_\alpha\rangle = -\tilde{v}_\alpha\|\nabla w_\alpha\|^2 \leq 0$$

For the nonlinear term, apply the Lipschitz condition:

$$|\langle N_\alpha(\tilde{u}_\alpha) - N_\alpha(\tilde{v}_\alpha), w_\alpha\rangle| \leq \|N_\alpha(\tilde{u}_\alpha) - N_\alpha(\tilde{v}_\alpha)\|\|w_\alpha\| \leq L_\alpha\|w_\alpha\|^2$$

Combine all terms:

$$\frac{1}{2}\frac{d}{dt}\|w_\alpha\|^2 \leq -\tilde{v}_\alpha\|\nabla w_\alpha\|^2 + L_\alpha\|w_\alpha\|^2$$

Neglecting the dissipative term to get an upper bound:

$$\frac{1}{2}\frac{d}{dt}\|w_\alpha\|^2 \leq 2L_\alpha\|w_\alpha\|^2$$

Applying Grönwall’s inequality, integrating over time gives:

$$\|w_\alpha(t)\|^2 \leq e^{2L_\alpha t}\|w_\alpha(0)\|^2$$

Therefore:

$$\|w_\alpha(t)\| \leq e^{L_\alpha t}\|w_\alpha(0)\|^2$$

In the nonlinear operator satisfies the Lipschitz condition and L_α is bounded, the fuzzy Navier-Stokes solution satisfies:

$$\|\tilde{u}_\alpha(t) - \tilde{v}_\alpha(t)\| \leq e^{L_\alpha t}\left\|\tilde{u}_{\alpha,0} - \tilde{v}_{\alpha,0}\right\|$$

Hence, the solution depends continuously on the initial fuzzy data so the 2D fuzzy Navier-Stokes equations are stable under the Lipschitz condition.

4.1 Theoretical Example:

To illustrate the stability result, consider a simple 2D fuzzy initial velocity field at α-level:

$$\tilde{u}_\alpha(x, y, 0) = [1-\alpha, 1+\alpha], \tilde{v}_\alpha(x, y, 0) = [0, 0.1\alpha]$$

where α ∈ [0,1] represents the membership level. Let the fuzzy viscosity be $\tilde{\nu}_\alpha = 0.1 + 0.05\alpha$ and assume the Lipschitz constant Lα=0.2 uniformly for all α.

Applying the stability bound from the main theorem:

$$\|\tilde{u}_\alpha(t) - \tilde{v}_\alpha(t)\| \le e^{L_\alpha t}\left\|\tilde{u}_{\alpha,0} - \tilde{v}_{\alpha,0}\right\|$$

we can estimate the bound at t = 1:

$$\|\tilde{u}_\alpha(1) - \tilde{v}_\alpha(1)\| \le e^{0.2}\|\tilde{u}_\alpha(0) - \tilde{v}_\alpha(0)\| \approx 1.22\|\tilde{u}_\alpha(0) - \tilde{v}_\alpha(0)\|$$

This example demonstrates that any small perturbation in the initial fuzzy data leads to a proportionally bounded deviation in the resulting fuzzy solution at all α-levels, confirming the theoretical stability result and illustrating how the α-cut representation captures the uncertainty at each level.

5. Discussion

The stability analysis presented in this paper demonstrates that the two-dimensional fuzzy Navier-Stokes equations exhibit continuous dependence on the fuzzy initial data under the Lipschitz condition. The main results expressed as:

$$\|\tilde{u}_\alpha(t) - \tilde{v}_\alpha(t)\| \le e^{L_\alpha t}\left\|\tilde{u}_{\alpha,0} - \tilde{v}_{\alpha,0}\right\|,$$

indicates that any perturbation in the initial fuzzy velocity field leads to a proportionally bounded deviation in the resulting fuzzy solution at all α −levels. This property ensures that the fuzzy solution remains well-behaved despite the uncertainties in the initial or boundary conditions.

From a physical perspective, the exponential factor $e^{L_\alpha t}$ captures the influence of the nonlinear advection term. If the Lipschitz constant L_α is small or if the viscous dissipation dominated (i.e., high fuzzy viscosity), the growth of perturbations is minimal, implying a asymptotic or practically uniform stability. Conversely, for large values of L_α, perturbations may growth more rapidly though the system remains bounded for finite time intervals.

The α −cut formulation allows these stability results to be interpreted at each level of uncertainty. Consequently, the overall fuzzy solution inherits the aforementioned stability properties by which the approach is exceptionally stable for uncertain or imprecise fluid dynamics problems, which could be problematic with classic deterministic estimation that fails to take into account fluctuations in parameters such as viscosity, density, or imposed external forces. In addition, the study lays the theoretical groundwork for numerical studies to come. For instance, finite derived stability bounds can serve as a guiding principle for the choice of time step and grid resolution when implementing fuzzy finite differential or finite volume schemes, so that the convergence and boundedness of the fuzzy computed solution can be ensured. Finally, while the present study focuses on the 2D incompressible case, the method has the potential to be expanded to include 3D flows, energy stability analysis with Lyapunov functionals, and hybrid stochastic-fuzzy models. These extensions could offer much finer grained understanding of the behavior of uncertain fluid systems for engineering and environmental applications.

6. Conclusions

In the process, stability analysis of two-dimensional fuzzy Navier-Stokes equations was performed with the α-cut representation and the Lipschitz condition. The system reformulated by an operator framework and Grönwall's inequality showed how a difference for any two fuzzy velocity solutions is always exponentially bounded in time. This is because the dependence of the fuzzy solution on initial data is continuous and hardly changes much, which indicates that the system is stable under uncertainty of physical parameters such as viscosity, density, and external forces. It shows that higher viscosity or small nonlinear Lipschitz constants provide stronger stability, while the α-cut approach gives a distinct picture of stability at each level of membership. The findings can serve as theoretical and numerical data for the analysis of uncertain fluid flow. Further progress to this step will take the generalization of stability analysis to three-dimensional fuzzy Navier-Stokes equations and introduce energy-based Lyapunov functionals obtaining more suitable stability estimates; in this way robust numerical schemes for fuzzy fluid dynamics are developed and used. The simulation and prediction of fluid behavior in engineering and environmental applications at unknown conditions would be expanded to a large extent.

References

[1] R. Ranjan, S. Unnikrishnan, and D. Gaitonde, A robust approach for stability analysis of complex flows using high-order Navier–Stokes solvers, *Journal of Computational Physics*, 403 (2020), 109076.

[2] Z. Y. Chen, Y. Meng, R. Y. Wang, and T. Chen, Systematic fuzzy Navier–Stokes equations for aerospace vehicles, *Aircraft Engineering and Aerospace Technology*, 94(3) (2022), 351–359.

[3] T. K. Sengupta, N. Sharma, and A. Sengupta, Non-linear instability analysis of the two-dimensional Navier–Stokes equation: The Taylor–Green vortex problem, *Physics of Fluids*, 30(5) (2018).

[4] H. Hashemi, R. Ezzati, N. M. Mikaeilvand, and M. Nazari, Fuzzy time-fractional advection-dispersion and Navier–Stokes equations: A comprehensive approach, *Iranian Journal of Fuzzy Systems*, 21(5) (2024), 105–119.

[5] M. Almheidat, H. Yasmin, M. Al Huwayz, R. Shah, and S. A. El-Tantawy, A novel investigation into time-fractional multi-dimensional Navier–Stokes equations within Aboodh transform, *Open Physics*, 22(1) (2024), 20240081.

[6] Z. A. Hasan and A. S. J. Al-Saif, A new approach to solving two-dimensional unsteady incompressible Navier–Stokes equations, *Journal of Applied Mathematics and Physics*, 10(10) (2022), 3218–3239.

[7] E. Carlson and A. Larios, Sensitivity analysis for the 2D Navier–Stokes equations with applications to continuous data assimilation, *Journal of Nonlinear Science*, 31(5) (2021), Article 84.

[8] X. G. Yang, Y. Qin, Y. Lu, and T. F. Ma, Dynamics of 2D incompressible non-autonomous Navier–Stokes equations on Lipschitz-like domains, *Applied Mathematics & Optimization*, 83(3) (2021), 2129–2183.

[9] F. Dannan, Lipschitz stability of nonlinear systems of differential equations. II. Liapunov functions, *Journal of Mathematical Analysis and Applications* (1989).

[10] H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei, and S. Noeiaghdam, Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions, *Journal of Vibration and Control*, 30(3–4) (2024), 632–647.

[11] J. Alzabut, A. G. M. Selvam, R. Dhineshbabu, S. Tyagi, M. Ghaderi, and S. Rezapour, A Caputo discrete fractional-order thermostat model with one and two sensors fractional boundary conditions depending on positive parameters by using the Lipschitz-type inequality, *Journal of Inequalities and Applications*, 2022(1) (2022), Article 56.

[12] Y. Y. Chen, R. J. Hsiao, and M. C. Huang, Fuzzy solutions for two-dimensional Navier–Stokes equations, *Journal of Mechanics*, 34(1) (2018), 1–10.

[13] L. Quartapelle, *Numerical Solution of the Incompressible Navier–Stokes Equations*, Vol. 113, Birkhäuser, (2013).

[14] S. Chakraverty, S. Tapaswini, and D. Behera, *Fuzzy Arbitrary Order System: Fuzzy Fractional Differential Equations and Applications*, John Wiley & Sons, (2016).

[15] K. Kashima and Y. Yamamoto, System theory for numerical analysis, *Automatica*, 43(7) (2007), 1156–1164.

[16] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24 (1987), 301–317.

[17] T. H. Grönwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Annals of Mathematics*, 20(4) (1919), 292–296.

[18] D. Dubois, *Fuzzy Sets and Systems: Theory and Applications*, Vol. 144, Academic Press, (1980).

[19] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, (2008).

[20] F. Tian, S. Liu, Z. Xu, and Q. Lei, Partial derivative and complete differential of binary intuitionistic fuzzy functions, *International Journal of Fuzzy Systems*, 19(2) (2017), 273–284.

[21] J. Heinonen, *Lectures on Lipschitz Analysis*, No. 100, University of Jyväskylä, (2005).

[22] Y. Feng, L. Li, J. G. Liu, and T. Tang, Some Grönwall inequalities for a class of discretizations of time fractional equations on nonuniform meshes, *SIAM Journal on Numerical Analysis*, 62(5) (2024), 2196–2221.

[23] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, Vol. 343, American Mathematical Society, (2024).