Differential Subordination and Superordination Results for a Subclass of Analytic Univalent Functions Using $A^{\mu}{}_{\theta}$ operator

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Abstract:

This paper investigates a new subclass of analytic univalent functions defined in the open unit disk $\mathrm{U} = \{\mathrm{z} \in \mathrm{C}: |\mathrm{z}| < 1\}$, by employing the generalized operator $A^{\mu}{}_{\theta,n} f(z)$.

Several sufficient conditions are established to determine differential subordination and superordination relationships for this subclass. In addition, corresponding sandwich-type results are derived based on these findings. The results obtained here extend and unify several previously known results for special operators and subclasses of analytic functions.

2020 Mathematics Subject Classification: 30C45
Keywords: Analytic function, Differential subordination, Differential superordination, Sandwich theorem, $A^{\mu}{}_{\theta,n} f(z)$ operator.

2. Introduction

let $S = S(U)$ represent the class of all functions that are analytic in U where $U = \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk. Define Let $S[a, n]$ be a subclass of the functions $f \in S$, given by the series:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \qquad n \in N, a \in \mathbb{C}. \tag{1}$$

We also assume that $\acute{S} \subset S$ where $\acute{S}$ is the subclass of analytic and univalent functions in U,of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Now, we assume that $f, g \in S$, and the function f is subordinate to g, or equivalently, g is superordinate to f, if there exists a Schwarz function W such that $f(z) = g(\mathrm{W}(z))$, where $\mathrm{W}(z)$ is analytic function in U with $|\mathrm{W}(z)| < 1$ and $\mathrm{W}(0) = 0, z \in U$,then one can say that $f \prec g$ or $f(z) \prec g(z)$ for $z \in U$. [13].

In addition, if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$

and $(U) \subset g\ (U)$. [13,17,18].

Definition 1[17]: Let $: \mathbb{C}^3 \times U \to \mathbb{C}$, and let $\natural(z)$ is univalent in U . If $\mathbb{P}(z)$ is analytic function in U and fulfills the second-order differential subordination:

$$\varphi(\mathbb{P}(z), z\,\mathbb{P}'(z), z^2\mathbb{P}''(z); z) \prec \natural(z), \tag{3}$$

then $\mathbb{P}(z)$ is said to be a solution of the differential subordination (3), and the univalent function $q(z)$ say it a dominant of the solution of differential subordination (3), or more simply a dominant, if $\mathbb{P}(z) \prec q(z)$ for each $\mathbb{P}(z)$ satisfying (3). A dominant function $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for each dominant $q(z)$ of (3) is called the *best dominant* of (3).

Definition 2[17]: Let $\mathbb{P}, \natural \in \acute{S}$ and $(r, s, t; z)$: $\mathbb{C}^3 \times U \to \mathbb{C}$. If $\mathbb{P}$ and $\varphi(\mathbb{P}(z), z\,\mathbb{P}'(z), z^2\mathbb{P}''(z); z)$ are univalent functions in U, and if $\mathbb{P}$ satisfies the second-order differential superordination:

$$\natural(z) \prec \varphi(\mathbb{P}(z), z\,\mathbb{P}'(z), z^2\mathbb{P}''(z); z), \tag{4}$$

then $\mathbb{P}$ is said to be a differential superordination solution (4). An analytic function $q(z)$, known as a subordinant of the solutions of differential superordination (4), or more simply , a subordinant , if $\mathbb{P} \prec q$ for each the functions $\mathbb{P}$ satisfying (4). If $\tilde{q}$ is univalent subordinant and that satisfy $q \prec \tilde{q}$ for each the subordinats q of (1.4), then is the best subordinat.

Many authors [1,2,3,10,17,20] obtained the necessary and sufficient conditions on the functions $\natural$, $\mathbb{P}$, and φ where by the following implication is true

$$\natural(z) \prec \varphi(\mathbb{P}(z), z\mathbb{P}'(z), z^2\mathbb{P}''(z); z),$$

Then

$$q(z) \prec \mathbb{P}(z) \tag{5}$$

Utilizing the outcomes Look [4,5,6,7,11,12,15,16,18,19,21] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1$and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Several authors [2,4,6,7,8,9] have also explored differential subordination and superordination results, along with sandwich theorems.

Let $f \in S$, $A^{\mu}{}_{\theta,n}\, f(z)$ defined the following generalized integral operator:

$$A^{\mu}{}_{\theta}\, f(z) = \frac{1-2\mu}{\left(\frac{1-2\mu}{2}\right)^{\frac{\theta+1}{2}} \Gamma\left(\frac{\theta}{2}+1\right)} \int_0^1 (\ln y^{2\mu-1})^{\frac{\theta-2}{2}}\, y\, f\left(z\sqrt{\ln y^{2\mu-1}}\right) dy, \tag{6}$$

where $\tau > 0, \lambda \geq 0, n \in N_0 = N_0 \cup \{0\}$.

For $f(z) \in A$ given by (2), we have the following expansion for $A^{\mu}{}_{\theta}\, f(z)$

$$A^{\mu}{}_{\theta}\, f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{\theta+n+1}{2}\right) \left(\frac{(1-2\mu)}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{\theta}{2}+1\right)} a_n z^n \tag{7}$$

From (7), we note that

$$z\left(A^{\mu}{}_{\theta,n}\, f(z)\right)' = (\theta+2)A^{\mu}{}_{\theta+2,n}\, f(z) - (\theta+1)A^{\mu}{}_{\theta,n}\, f(z) \tag{8}$$

The specific goal of this research is to find sufficient conditions for certain normalized analytic function f to satisfy:

$$q_1(z) \prec \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\gamma} \prec q_2(z),$$

and

$$q_1(z) \prec \left[\frac{A^{\mu}{}_{\theta+2,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\gamma} \prec q_2(z),$$

wherever q_1 and q_2 are provided univalent functions in U with $q_1(0) = q_2(0) = 1$.

In this paper, we will derive Some sandwich theorems with the operator $A^{\mu}{}_{\theta,n}\, f(z)$.

Recent studies (2020–2024) have extensively explored conditions under which analytic and univalent functions satisfy differential subordination and superordination relations, especially in the context of generalized linear and fractional operators (see [1–3, 10, 17, 20, 22–26]). These investigations have unified several classical results in geometric function theory and have led to the development of sandwich-type theorems, as well as improved growth and distortion estimates (see [4–9, 11–19, 21]).

This work contributes to the existing literature by employing the generalized operator $A^{\mu}{}_{\theta,n}\, f(z)$ to derive sufficient conditions for differential subordination, superordination, and sandwich-type results for a newly defined subclass of analytic univalent functions. In doing so, it extends and generalizes recent results, offering a broader framework that connects classical findings with new subclasses generated by this operator.

2. preliminaries

To prove the main results, we will need the following lemmas and definitions.

Definition 1 [17]: Let Q denote the class of all functions q that are analytic and injective on $\overline{U}|E(q)$, where $\overline{U} = U \cup \{z \in \partial U\}$, and

$$E(q) = \left\{v \in \partial U : \lim_{z\to\varepsilon} q(z) = \infty\right\}$$

Further, let $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U \backslash E(q)$. The subclass of Q where $q(0) = a$ is denoted $Q(a)$, and specifically,

$$Q(0) = Q_0, and\ Q(1) = Q_1 = \{q \in Q : q(0) = 1\}.$$

Lemma 1 [18]: Suppose that the function q is a convex univalent in U let $\lambda \in \mathbb{C}$, $B \in \mathbb{C}|\{0\}$ if

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re(\frac{\lambda}{B})\right\} \tag{9}$$

and if P is analytic in U and satisfies the subordination condition

$$\lambda\mathbb{P}(z) + Bz\mathbb{P}'(z) \prec \lambda q(z) + Bzq'(z), \tag{10}$$

then $\mathbb{P} \prec q$ and q is the best dominant of (10).

Lemma 2 [5]: Let q be univalent in U, and let φ and θ be analytic in the domain D containing $q(U)$ with $\varphi(e) \neq 0$, when $W \in q(U)$. Define

$$h(z) = zq'(z)\varphi\big(q(z)\big)\ , h(z) = \theta\big(q(z)\big) + Q(z).$$

Suppose that $Q(z)$ is starlike univalent in U, and that

$$Re\left\{\frac{zh\prime(z)}{Q(z)}\right\} > 0\ , z \in U$$

If P is analytic in U, with $\mathbb{P}(0) = q(0)$, $\mathbb{P}(U) \subseteq D$, and satisfies the subordination condition

$$\theta\big(\mathbb{P}(z)\big) + z\mathbb{P}'(z)\varphi\big(\mathbb{P}(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\varphi\big(q(z)\big), \tag{11}$$

and

$$h(z) \prec \varphi(hq(z), z\ \mathbb{P}'(z), z^2\mathbb{P}''(z); z)$$

then $\mathbb{P} \prec q$ and q is the best dominant of (11).

Lemma 3 [18]: Suppose that q is convex univalent in U and let $B \in \mathbb{C}$, with $Re(B) > 0$. If $\mathbb{P} \in H[q(0), 1] \cap Q$ and $\mathbb{P}(z) + Bz\mathbb{P}'(z)$ is univalent in U, then

$$q(z) + Bzq'(z) \prec \mathbb{P}(z) + Bz\mathbb{P}'(z), \tag{12}$$

which implies that $q \prec \mathbb{P}$ and q is the best subordinant of (12).

Lemma 4 [12]: Let $q(z)$ be a convex univalent function in the unit disk U , and let φ and θ be analytic in the domain D containing $q(U)$. Suppose that:

$$Re\left\{\frac{\theta'\big(q(z)\big)}{\varphi\big(q(z)\big)}\right\} > 0\ , z \in U,$$

and that $Q(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.

If $\mathbb{P} \in S[\mathfrak{q}(0), 1] \cap Q$,with $\mathbb{P}(U) \subset D$,and satisfies the condition $\theta\big(\mathbb{P}(z)\big) + z\mathbb{P}'(z)\varphi(\mathbb{P}(z))$ is univalent in U

and

$$\big(\mathfrak{q}(z)\big) + z\mathfrak{q}'(z)\varphi(\mathfrak{q}(z)) \prec \theta\big(\mathbb{P}(z)\big) + z\mathbb{P}'(z)\varphi(\mathbb{P}(z)), \tag{13}$$

then $\mathfrak{q} \prec \mathbb{P}$ and $\mathfrak{q}$ is the best subordinant of (12).

3. Differential Subordination Results

We Present a few differential subordination results by using the $A^{\mu}{}_{\theta,n}\, f(z)$.

Theorem 1. Suppose that $\mathfrak{q}$ be a convex univalent function in U with $\mathfrak{q}(0) = 1, \eta > 0, 0 \neq v \in \mathbb{C}$, and suppose that $\mathfrak{q}$ satisfies:

$$Re\left\{1 + \frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right\} > \max\left\{0, -Re\left(\frac{i}{v}\right)\right\}. \tag{14}$$

If $f \in \acute{S}$ satisfies the subordination condition:

$$\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}\left[(\theta+2)\left(\frac{\left(A^{\mu}{}_{\theta+2,n}\, f(z)\right)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right)\right]$$

$$\prec \mathfrak{q}(z) + \frac{i}{v} z\mathfrak{q}'(z)\,, \tag{15}$$

then

$$\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta} \prec \mathfrak{q}(z) \tag{16}$$

and $\mathfrak{q}$ is the best dominant of (15).

Proof. We shall define the function $\mathbb{P}$ by

$$\mathbb{P}(z) = \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}, \tag{17}$$

then the function $\mathbb{P}(z)$ is analytic and $\mathbb{P}(0) = 1$, therefore, differentiating (16) with respect to(z) and using the identity (8), we obtain

$$\frac{z\mathbb{P}'(z)}{\mathbb{P}(z)} = \eta\left[\frac{z\left(A^{\mu}{}_{\theta,n}\, f(z)\right)'}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right] \tag{18}$$

Hence

$$\frac{z\mathbb{P}'(z)}{\mathbb{P}(z)} = \eta\left[(\theta+2)\left(\frac{\left(A^{\mu}{}_{\theta+2,n}\, f(z)\right)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right)\right].$$

Therefore,

$$\frac{z\mathbb{P}'(z)}{\eta} = \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta} * \left[(\theta+2)\left(\frac{\left(A^{\mu}{}_{\theta+2,n}\, f(z)\right)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right)\right].$$

The subordination (14) from the hypothesis becomes

$$\mathbb{P}(z) + \frac{i}{v} z\mathbb{P}'(z) \prec \mathfrak{q}(z) + \frac{i}{v} z\mathfrak{q}'(z).$$

An application of lemma (2.1) with $Ƀ = \frac{i}{v}$ and $v = 1$, we obtain (15).

putting $\mathfrak{q}(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (3.1), we obtain the following corollary:

Corollary1. Let $\eta > 0, 0 \neq \varepsilon \in \mathbb{C}\backslash\{0\}$ and

$$Re\left\{1 + \frac{2z}{1-z}\right\} > \max\left\{0, -Re\left(\frac{i}{v}\right)\right\}.$$

If $f \in S$ satisfies the subordination condition:

$$\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}\left[(\theta+2)\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right)\right] \prec \left(\frac{1 - z^2 + 2\frac{\varepsilon}{\gamma} z}{(1-z)^2}\right),$$

then

$$\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta} \prec \left(\frac{1+z}{1-z}\right)$$

and $\mathfrak{q}(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 2. Let $\mathfrak{q}$ be a convex univalent function in U with

$\mathfrak{q}(0) = 1, \mathfrak{q}'(z) \neq 0\ (z \in U)$ and assume that $\mathfrak{q}$ satisfies:

$$Re\left\{1 + \frac{i}{v}\left(\mathfrak{q}(z)\right)^{i} + \frac{i-1}{v}\left(\mathfrak{q}(z)\right)^{i-1} - z\frac{\mathfrak{q}'(z)}{\mathfrak{q}(z)} + z\frac{\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right\} > 0, \tag{19}$$

where $i \in \mathbb{C}, v \in \mathbb{C}\backslash\{0\}$and $z \in U$. Suppose that $z\frac{\mathfrak{q}'(z)}{\mathfrak{q}(z)}$ is starlike univalent in U. If $f \in S$ satisfies

$$\psi(\eta,\mu,\theta,i,v;z) \prec (1+\mathrm{q}(z))\mathrm{q}(z)^{i-1} + vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)}, \tag{20}$$

Where

$$\psi(\eta,\mu,\theta,n,v;z) = \left[\frac{A^{\mu}{}_{\theta+1,n}\,f(z)}{A^{\mu}{}_{\theta,n}\,f(z)}\right]^{\eta i} + \left[\frac{A^{\mu}{}_{\theta+1,n}\,f(z)}{A^{\mu}{}_{\theta,n}\,f(z)}\right]^{\eta(i-1)}$$

$$+\eta(\theta+2)\left(\frac{\left(A^{\mu}{}_{\theta+1,n}\,f(z)\right)}{A^{\mu}{}_{\theta-1,n}\,f(z)} - \frac{\left(A^{\mu}{}_{\theta+2,n}\,f(z)\right)}{A^{\mu}{}_{\theta,n}\,f(z)}\right), \tag{21}$$

then

$$\left[\frac{A^{\mu}{}_{\theta+1,n}\,f(z)}{A^{\mu}{}_{\theta,n}\,f(z)}\right]^{\eta} \prec \mathrm{q}(z)\,, \tag{22}$$

and q is the best dominant of (20).

Proof. Consider a function $\mathbb{P}$ by

$$\mathbb{P}(z) = \left[\frac{A^{\mu}{}_{\theta+1,n}\,f(z)}{A^{\mu}{}_{\theta,n}\,f(z)}\right]^{\eta}, \tag{23}$$

Then the function $\mathbb{P}(z)$ is analytic in U and $\mathbb{P}(0)=1$, differentiating (23), with respect to z and using the identity (8), we obtain

$$\frac{z\mathbb{P}'(z)}{\mathbb{P}(z)} = \left[\eta(\theta+2)\left(\frac{\left(A^{\mu}{}_{\theta+1,n}\,f(z)\right)}{A^{\mu}{}_{\theta-1,n}\,f(z)} - \frac{\left(A^{\mu}{}_{\theta+2,n}\,f(z)\right)}{A^{\mu}{}_{\theta,n}\,f(z)}\right)\right].$$

By setting

$\theta(e) = (1+e)e^{i-1} and\ \phi(e) = \frac{i}{v}\ , v \neq 0$

we see that $\theta(e)$ is analytic in $\mathbb{C}$ and $\theta(e)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\theta(e) \neq 0, \theta \in \mathbb{C}|\{0\}$. Also, we obtain

$$Q(z) = zq(z)\,\phi\big(zq(z)\big) = vz\frac{q'(z)}{q(z)}$$

and

$$\mathrm{h}(z) = \theta\big(\mathrm{q}(z)\big) + Q(z) = \big(1+\mathrm{q}(z)\big)\mathrm{q}(z)^{i-1} + vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)}.$$

It is obvious that $Q(z)$ is starlike univalent in U, we have

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > Re\left\{\frac{i}{\mathrm{v}}q(z)^{i}q'(z) + \frac{i-1}{\mathrm{v}}q(z)^{i-1}q'(z)\right\} > 0$$

Using a simple calculation, we get

$$\psi(\eta,\mu,\theta,n,v;z) \prec \big(1+q(z)\big)q(z)^{i-1} + vz\frac{q'(z)}{q(z)}, \tag{24}$$

where $\psi(\eta,\mu,\theta,n,v;z)$ is given by (19).

From (20) and (24), we have

$$\begin{gathered}\big(1+\mathbb{P}(z)\big)(\mathbb{P}(z))^{i-1} + vz\frac{\mathbb{P}'(z)}{\mathbb{P}(z)} \prec \\ \big(1+q(z)\big)\big(q(z)\big)^{i-1} + vz\frac{q'(z)}{q(z)}.\end{gathered} \tag{25}$$

Therefore, by Lemma (2.2), we get $\mathbb{P}(z) \prec q(z)$. By using (23), we get the result.

make up $q(z) = \left(\frac{1+Az}{1+Bz}\right)$, $-1 \le B < A \le 1$ in Theorem(3.2), we obtain the following:

Corollary 2 Let $-1 \le B < A \le 1$ and

$$Re\left\{\frac{i}{v}\left(\frac{1+Az}{1+Bz}\right)^{i} + \frac{i-1}{v}\left(\frac{1+Az}{1+Bz}\right)^{i-1} + \frac{1+Bz(4+3Az)}{(1+Bz)(1+Az)}\right\} > 0,$$

where $v \in \mathbb{C}\backslash\{0\}$ and $z \in U$, if $f \in \acute{S}$ satisfies:

$$\psi(\eta,\mu,\theta,i,v;z) \prec \left[\left[1+\left(\frac{1+Az}{1+Bz}\right)\right]\left(\frac{1+Az}{1+Bz}\right)^{i-1} + vz\frac{A-B}{(1+Az)(1+Bz)}\right],$$

and $\psi(\eta,\mu,\theta,i,v;z)$is given by (3.8),

then

$$\left[\frac{A^{\mu}{}_{\theta+1,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\eta} \prec \left(\frac{1+Az}{1+Bz}\right)$$

and $q(z) = \left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

4. Differential Superordination Results

Theorem 3. Let q be a convex univalent function in U with

$q(0) = 1$, $\eta > 0$ and $Re\{v\} > 0$. Let $f \in \acute{S}$ satisfies :

$$\left[\frac{A^{\mu}{}_{\theta+1,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\eta} \in S[q(0),1] \cap Q \text{ and } \quad (4.1)$$

and

$\left[(\theta+2)\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n} f(z)}{A^{\mu}{}_{\theta,n} f(z)}-1\right)\right]$ be univalent in U. If(4.2)

$$\mathrm{q}(z)+\frac{i}{v} z\mathrm{q}'(z) \prec \left[(\theta+2)\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n} f(z)}{A^{\mu}{}_{\theta,n} f(z)}-1\right)\right] + \left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}, \tag{26}$$

then

$$\mathrm{q}(z) \prec \left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}, \tag{27}$$

and q is the best subordinant of (26).

Proof. Define the function $\mathbb{P}$ by

$$\mathbb{P}(z)=\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}. \tag{28}$$

Differentiating (28) with respect to z , we get

$$\frac{z\mathbb{P}'(z)}{\mathbb{P}(z)}=\eta\left[\frac{z\left(A^{\mu}{}_{\theta,n} f(z)\right)'}{A^{\mu}{}_{\theta,n} f(z)}-1\right], \tag{29}$$

We use (1.8) with some simplification from (29), we get

$$(\theta+2)\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n} f(z)}{A^{\mu}{}_{\theta,n} f(z)}-1\right)+\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}=\mathbb{P}(z)+\frac{i}{v} z\mathbb{P}'(z).$$

by using Lemma (**2.3**) , we get the desired result.

putting $\mathrm{q}(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem **4.1**, we obtain the subsequent corollary:

Corollary 3. Let $\eta>0$ and $Re\{v\}>0$. If $f \in \mathrm{q}$ satisfies

$$\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta} \in S[\mathrm{q}(0),1] \cap Q$$

and

$\left[(\theta+2)\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n} f(z)}{A^{\mu}{}_{\theta,n} f(z)}-1\right)\right]+\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}$ be univalent in U. If

$\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right) \prec (\theta+2)\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n} f(z)}{A^{\mu}{}_{\theta,n} f(z)}-1\right)+\left[\frac{A^{\mu}{}_{\theta,n} f(z)}{z}\right]^{\eta},$

then $\left(\frac{1+z}{1-z}\right) \prec \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}$,

and $\mathrm{q}(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4. Let q be a convex univalent function in U with

$\mathrm{q}(0) = 1, \mathrm{q}'(z) \neq 0$ and Suppose that q satisfies:

$$Re\left\{\frac{i}{v}\left(\mathrm{q}(z)\right)^{i}\mathrm{q}'(z) + \frac{i-1}{v}\left(\mathrm{q}(z)\right)^{i-1}\mathrm{q}'(z)\right\} > 0, \tag{30}$$

where $i \in \mathbb{C}, v \in \mathbb{C}\backslash\{0\}$ and $z \in U$.

Let $z\frac{\mathrm{q}'(z)}{\mathrm{q}(z)}$ is starlike univalent function in U. Let $f \in \acute{S}$ satisfies

$$\left[\frac{A^{\mu}{}_{\theta+1,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\eta} \in H[\mathrm{q}(0), 1] \cap Q,$$

and $\psi(\eta, \mu, \theta, i, v; z)$ is univalent function in U, where $\psi(\eta, \mu, \theta, i, v; z)$ is given by (21). If

$$\left(1 + \mathrm{q}(z)\right)\left(\mathrm{q}(z)\right)^{i-1} + vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)} \prec \psi(\eta, \mu, \theta, i, v; z), \tag{31}$$

then

$$\left(1 + \mathrm{q}(z)\right)\left(\mathrm{q}(z)\right)^{i-1} + vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)} \prec \psi(\eta, \mu, \theta, i, v; z), \tag{32}$$

and q is the best subordinant of (31).

proof. Consider a function $\mathbb{P}$ by

$$\left(1 + \mathrm{q}(z)\right)\left(\mathrm{q}(z)\right)^{i-1} + vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)} \prec \psi(\eta, \mu, \theta, i, v; z), \tag{33}$$

Differentiating(4.10) with respect to z , we obtain

$$\frac{z\mathbb{P}'(z)}{\mathbb{P}(z)} = \left[\eta(\theta + 2)\left(\frac{\left(A^{\mu}{}_{\theta+1,n}\, f(z)\right)}{A^{\mu}{}_{\theta-1,n}\, f(z)} - \frac{\left(A^{\mu}{}_{\theta+2,n}\, f(z)\right)}{A^{\mu}{}_{\theta,n}\, f(z)}\right)\right].$$

By setting $\theta(e) = (1 + e)e^{i-1} and\ \phi(e) = \frac{i}{v}$, $v \neq 0$.

It is simple to see that $\theta(e)$ is analytic in $\mathbb{C}$ *and* $\phi(e)$ is analytic in $\mathbb{C}/\{0\}$ and that $\phi(e) \neq 0$, $e \epsilon \mathbb{C}/\{0\}$ as well , if we let

$$Q(z) = z\mathrm{q}'(z)\varphi(\mathrm{q}(z)) = vz\frac{\mathrm{q}'(z)}{\mathrm{q}(z)},$$

we see that $Q(z)$ is starlike univalent function in U,

$$Re\left\{\frac{\theta'(\mathrm{q}(z))}{\varphi(\mathrm{q}(z))}\right\} = Re\left\{\frac{i}{v}(\mathrm{q}(z))^{i}\mathrm{q}'(z) + \frac{i-1}{v}(\mathrm{q}(z))^{i-1}\mathrm{q}'(z)\right\} > 0.$$

Using a simple calculation, we obtain

$$\psi(\eta,\mu,\theta,i,v;z) = (1+\mathbb{P}(z))(\mathbb{P}(z))^{i-1} + vz\frac{\mathbb{P}'(z)}{\mathbb{P}(z)}, \tag{34}$$

where $\psi(\eta,\mu,\theta,i,v;z)$is given by (3.8).

We have from (31) and (34)

$$\psi(\eta,\mu,\theta,i,v;z) = (1+\mathbb{P}(z))(\mathbb{P}(z))^{i-1} + vz\frac{\mathbb{P}'(z)}{\mathbb{P}(z)}, \tag{35}$$

Therefore, by Lemma(2.4), we get $\mathrm{q}(z) \prec \mathbb{P}(z)$.

5. Sandwich Results

Theorem 5. Let q_1 be a convex univalent function in U with

$\mathrm{q}_1(0) = 1$, $\eta > 0$ and $Re\{\varepsilon\} > 0$ and let q_2 be univalent function in U, $\mathrm{q}_2(0) = 1$ and satisfies (14). Let $f \in s$ satisfies:

$$\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta} \in S[1,1] \cap Q$$

and

$(\theta + 2) \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right) + \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}$ be univalent in U. If

$$\mathrm{q}_1(z) + \frac{i}{v}z\mathrm{q}_1{}'(z) \prec (\theta+2)\left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}\left(\frac{A^{\mu}{}_{\theta+2,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)} - 1\right) + \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta}$$
$$\prec \mathrm{q}_2(z) + \frac{i}{v}z\mathrm{q}_2{}'(z),$$

then

$$\mathrm{q}_1(z) \prec \left[\frac{A^{\mu}{}_{\theta,n}\, f(z)}{z}\right]^{\eta} \prec \mathrm{q}_2(z),$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 6. Let q_1 be a convex univalent function in U with $\mathrm{q}_1(0) = 1$ and satisfies (30) . Let q_2 be univalent function in U with $\mathrm{q}_2(0) = 1$ and satisfies (19). Let $f \in \acute{S}$ satisfies

$$\left[\frac{A^{\mu}{}_{\theta+1,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\eta} \in S[1,1] \cap Q,$$

and $\psi(\eta, \mu, \theta, i, v; z)$ is univalent in U, where $\psi(\eta, \mu, \theta, i, v; z)$ is given by (21).If

$$\big(1 + \mathrm{q}_1(z)\big)\big(\mathrm{q}_1(z)\big)^{i-1} + vz\frac{\mathrm{q}_1{}'(z)}{\mathrm{q}_1(z)} \prec \psi(\eta, \mu, \theta, i, v; z)$$

$$\prec \big(1 + \mathrm{q}_2(z)\big)\big(\mathrm{q}_2(z)\big)^{i-1} + vz\frac{\mathrm{q}_2{}'(z)}{\mathrm{q}_2(z)},$$

then

$$\mathrm{q}_1(z) \prec \left[\frac{A^{\mu}{}_{\theta+1,n}\, f(z)}{A^{\mu}{}_{\theta,n}\, f(z)}\right]^{\eta} \prec \mathrm{q}_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Conclusion

In this work, we studied a subclass of analytic univalent functions associated with the generalized operator $A^{\mu}{}_{\theta,n}\, f(z)$. Using differential subordination and superordination techniques, we obtained sufficient conditions for normalized analytic functions and established corresponding sandwich-type results. These findings extend previous works and highlight the effectiveness of the operator $A^{\mu}{}_{\theta,n}$ in generating new results within geometric function theory.

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